Principal Component Regression

Nate Wells

Math 243: Stat Learning

November 16th, 2020

Outline

In today's class, we will. . .

- Discuss Principal Component Analysis as a means of dimensionality reduction for regresion
- Implement PCR in R

Section 1

Principal Component Regression

Dimensionality Reduction

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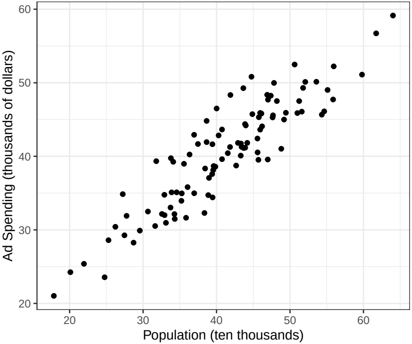
- But dropping variables completely loses possible valauable information.
- Instead, we can combine variables into new ones that adequately describe the variance in the data, and drop those that have limited utility in explaining that variance.

PCA Overview

Consider the relationship between campaign ad spending and population size for 100 cities:



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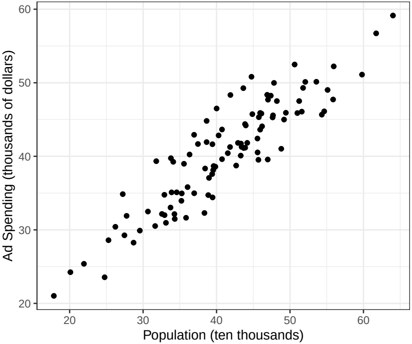

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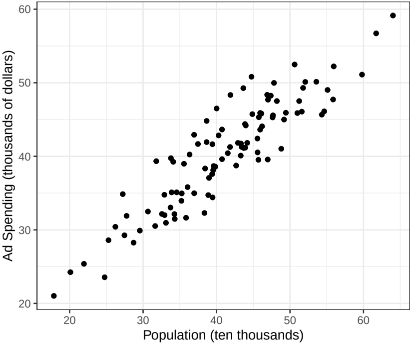

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```
##      sd_Pop    sd_Ad
## 1 8.981994 7.418227
```

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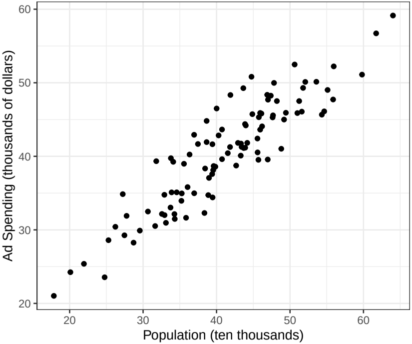

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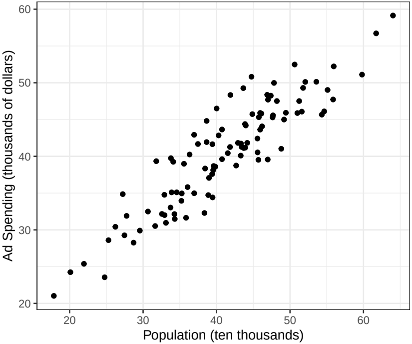

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```
##         R_sq
## 1 0.8238886
```

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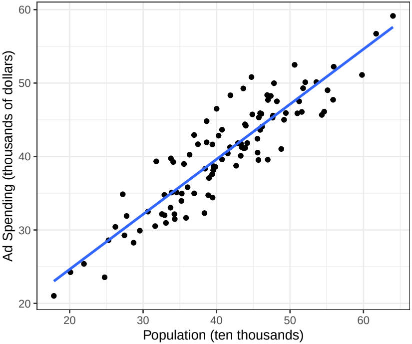
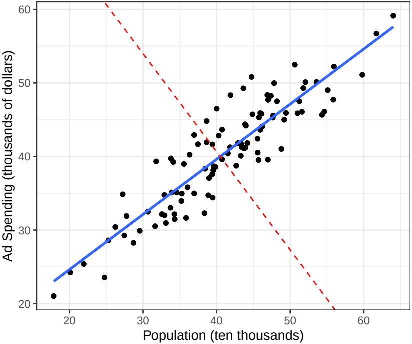


PCA Overview

How much variation occurs perpendicular to this line?



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We can express the first principal component as a linear combination of the centered predictors $X_i - \bar{X}_i$, where $\phi_{i1} \in \mathrm{R}$ and $\phi_{11}^2 + \cdots + \phi_{p1}^2 = 1$:

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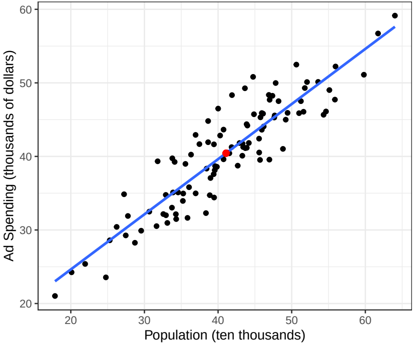
- Alternatively, we could express Z_1 as an affine linear combination of the predictors themselves (affine meaning including a constant term)

PCA Example

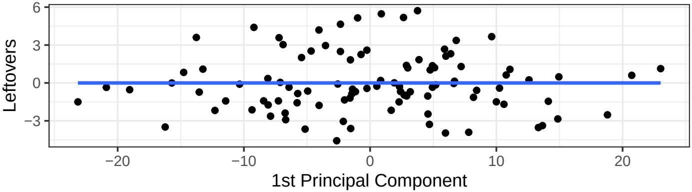
The first principal component



$$Z_1 = 0.8(\mathrm{Pop} - 41.1) + 0.6(\mathrm{Ad} - 40.4)$$

PCA Example

What is leftover?



Other Principal Components

In general, if we have p predictors, we can compute p distinct principal components: $Z_1, Z_2, \ldots, Z_p$.

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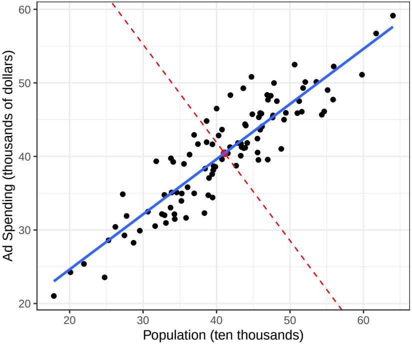
For the case when $p = 2$, the 2nd principal component corresponds to the line perpendicular to the line for the 1st principal component.

Generally, the kth principal component is obtained by finding a linear combination of centered variables that is uncorrelated with all previous principal components, and has the largest variance subject to this constraint.

PCA Example

The second principal component



$$Z_2 = 0.6(\text{Pop} - 41.1) - 0.8(\text{Ad} - 40.4)$$

Principal Comoponent Regression

The PCR approach to linear regression constructs the first M principal components $Z_1, \ldots, Z_M$ of a data set with p predictors (so $M \leq p$), and then uses these as predictors in a linear regression model.

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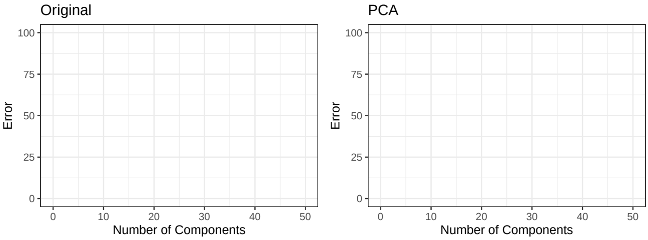
- Goal: Use a small number of predictors which explain most of the variability in the data set, as well as their relationship to the response.

In general, PCR tends to produce linear models with higher accuracy than models fit with the original predictors.



Principal Component Regression in R

We can use the `pcr` function in the `pls` library to quickly perform PCR in R.

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The `Hitters` data set from the `ISLR` package contains `Salary` and 18 other predictors for 263 baseball players

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set.seed(1)
library(pls)
my_pcr <- pcr( Salary ~ ., data = Hitters, scale = T, validation = "CV")
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- Setting `scale = T` standardizes each predictor
- Setting `validation = "CV"` causes `pcr` to compute the 10-fold CV error for each value of M (number of principal components used)

PCR Results

```
summary(my_pcr)
```

```
## Data:    X dimension: 263 19
##  Y dimension: 263 1
## Fit method: svdpc
## Number of components considered: 19
##
## VALIDATION: RMSEP
## Cross-validated using 10 random segments.
##        (Intercept)  1 comps  2 comps  3 comps  4 comps  5 comps  6 comps
## CV             452    352.5    351.6    352.3    350.7    346.1    345.5
## adjCV          452    352.1    351.2    351.8    350.1    345.5    344.6
##        7 comps  8 comps  9 comps  10 comps  11 comps  12 comps  13 comps
## CV       345.4    348.5    350.4     353.2     354.5     357.5     360.3
## adjCV    344.5    347.5    349.3     351.8     353.0     355.8     358.5
##        14 comps  15 comps  16 comps  17 comps  18 comps  19 comps
## CV        352.4     354.3     345.6     346.7     346.6     349.4
## adjCV     350.2     352.3     343.6     344.5     344.3     346.9
##
## TRAINING: % variance explained
##         1 comps  2 comps  3 comps  4 comps  5 comps  6 comps  7 comps  8 comps
## X         38.31    60.16    70.84    79.03    84.29    88.63    92.26    94.96
## Salary    40.63    41.58    42.17    43.22    44.90    46.48    46.69    46.75
##         9 comps  10 comps  11 comps  12 comps  13 comps  14 comps  15 comps
## X         96.28     97.26     97.98     98.65     99.15     99.47     99.75
## Salary    46.86     47.76     47.82     47.85     48.10     50.40     50.55
##         16 comps  17 comps  18 comps  19 comps
## X          99.89     99.97     99.99    100.00
## Salary     53.01     53.85     54.61     54.61
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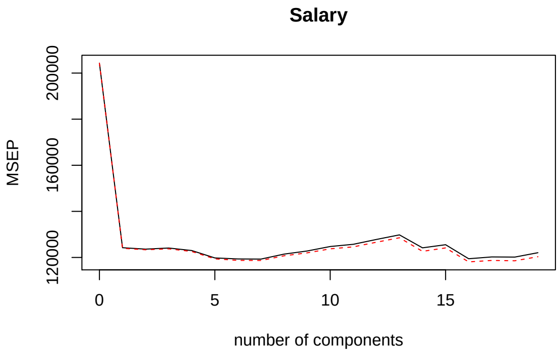
- Note: `pcr` reports RSE, so values need to be squared to get MSE.

Validation Plot

```r
validationplot(my_pcr, val.type = "MSEP")
```



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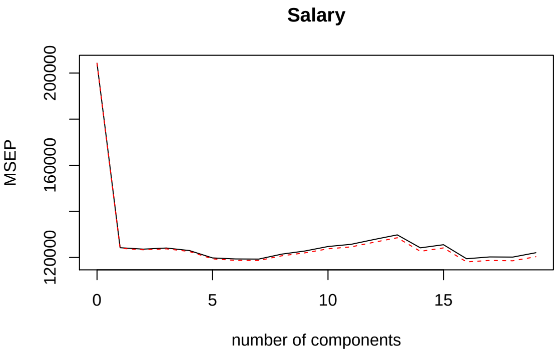

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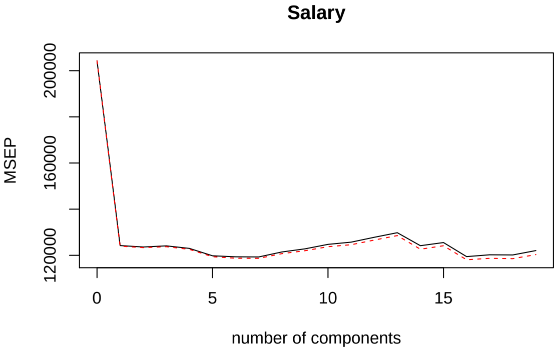

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- However, a relatively low CV error is also obtained at $M = 6$, suggesting fewer components are sufficient

Comparative Performance

Live coding. A .Rmd file will be available on course website after class