Assessing Model Accuracy

Nate Wells

Math 243: Stat Learning

September 9th, 2020

Outline

In today's class, we will...

- Discuss theoretical foundation for linear regression
- Assess accuracy of simple linear models
- Implement simple linear regression in R

Foundations

Linear Regression

- Suppose we have one or more predictors $(X_1, X_2, \ldots, X_p)$ and a *quantitative* respone variable Y, and that
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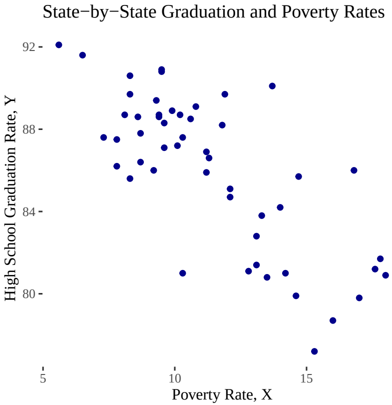
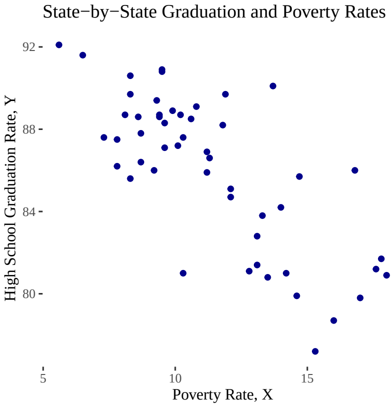
- So we are **estimating** an **approximation** to a relationship between response and predictors.

SLR Review


State−by−State Graduation and Poverty Rates
High School Graduation Rate, Y
92
88
84
80
5
10
15
Poverty Rate, X

SLR Review



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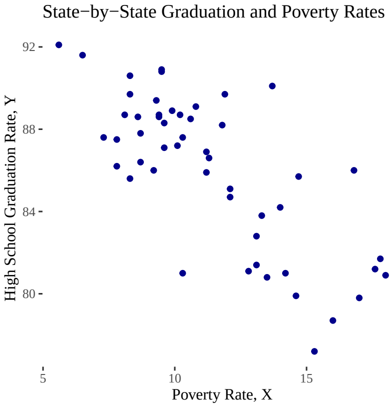

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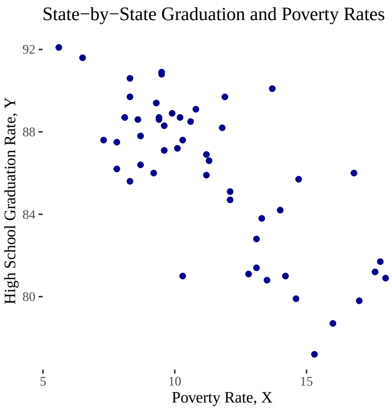

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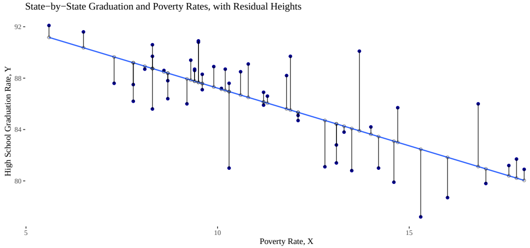

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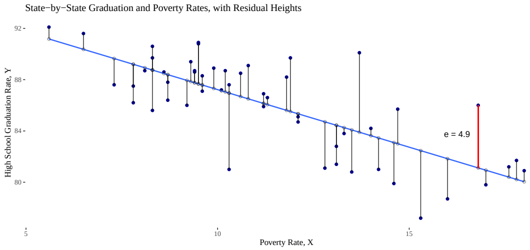
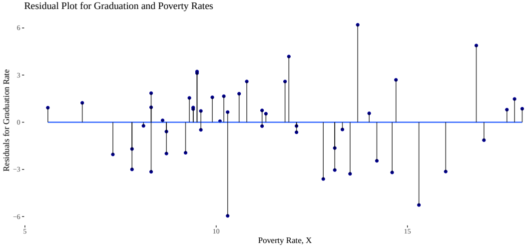


- D.C.'s residual is

$$e = y - \hat{y} = 86 - 81.1 = 4.9$$

Residual Plot

- To visualize the degree of accuracy of a linear model, we use residual plots:



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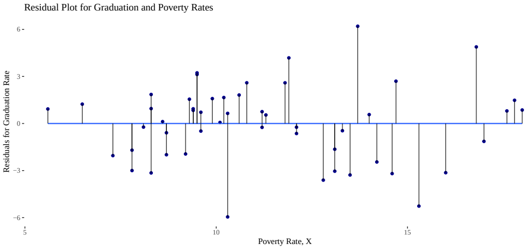

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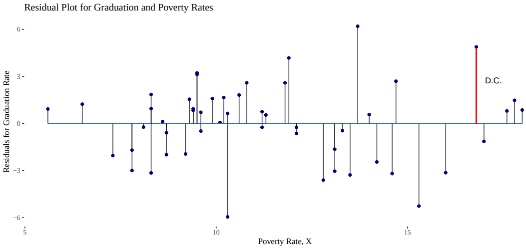

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- Using calculus, we can show that RSS is minimized when

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Assessing Accuracy

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- **Tools**: confidence intervals, hypothesis tests
- **The Problems**: Our model will change if built using a different random sample. So in addition to estimates, we need to know about variability

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- And where $\mathrm{SE}(\hat{\theta})$ is the standard error of $\hat{\theta}$, or the standard deviation of the sampling distribution

Common Regression Assumptions

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4. The errors are normally distributed: $\epsilon \sim N(0, \sigma^2)$

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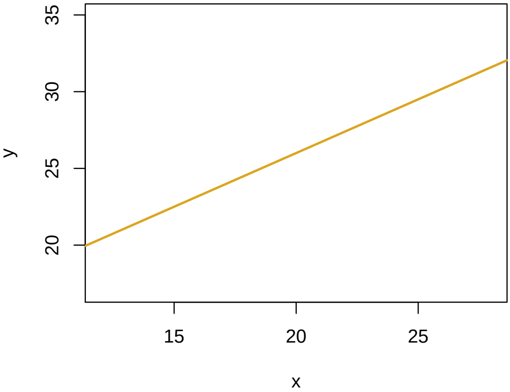

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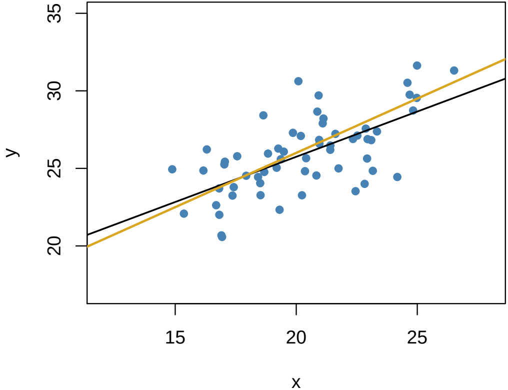

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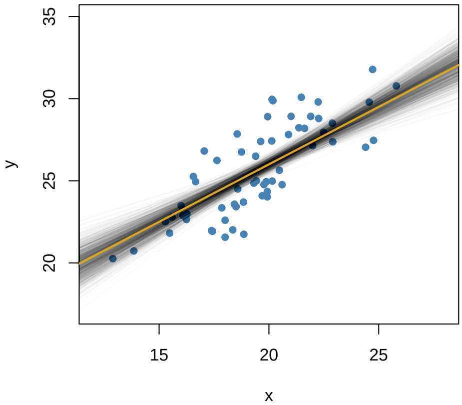

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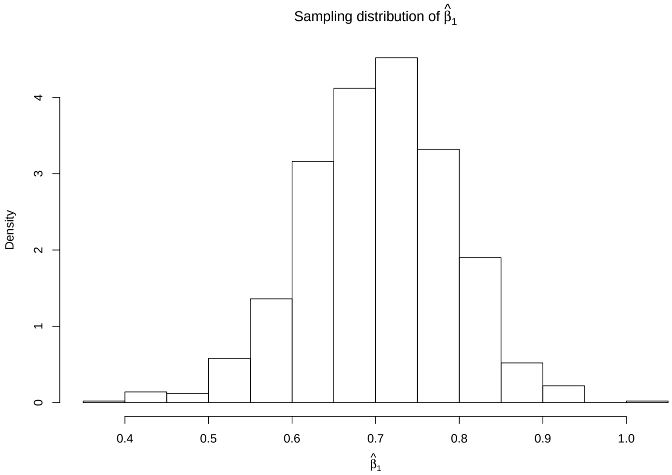

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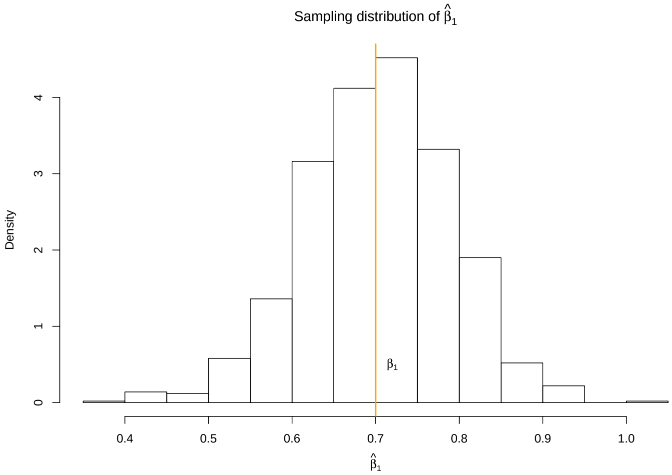

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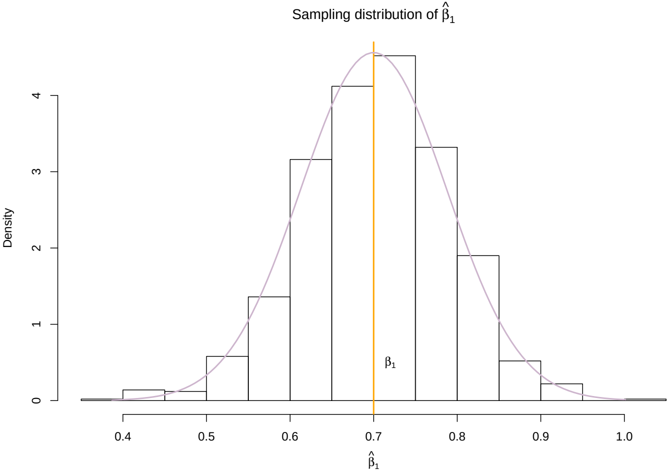

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3. $\hat{\beta}_1 | X \sim N(\beta_1, \frac{\sigma^2}{SXX})$.

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Interpretation We are *95% confident* that the true slope between x and y lies between LB and UB.

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T will be t distributed with $n - 2$ degrees of freedom and with $SE(\hat{\beta}_1)$ calculated the same as in the CI.

Inference for $\hat{\beta}_0$

Often less interesting (but not always!). You use the t-distribution again but with a different SE.